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Physical Components, Coordinate Components, and the Speed of Light

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Abstract

For generalized coordinate systems, the numerical values of vector and tensor components do not generally equal the physical values, i.e., the values one would measure with standard physical instruments. Hence, calculating physical components from coordinate components is important for comparing experiment with theory. Surprisingly, however, this calculational method is not widely known among physicists, and is rarely taught in relativity courses, though it is commonly employed in at least one other field (applied mechanics.) Different derivations of this method, ranging from elementary to advanced level, are presented. The result is then applied to clarify the oftentimes confusing issue of whether or not the speed of light in non-inertial frames is equal to c.

1 INTRODUCTION

Although orthonormal (Cartesian and Lorentz) coordinate systems are the easiest to understand, they are not always the easiest to use to solve a given problem. An approximately infinite straight line wire antenna, for example, is far easier to analyze in a cylindrical coordinate system than a Cartesian one. More complicated coordinate systems (examples include spherical and toroidal systems) are also often used to make geometrically complex problems tractable. And in the most general case, one can imagine, as Einstein did with his coordinate "mollusk", a collection of grid lines spread throughout space in a somewhat haphazard manner, crossing one another in no particular pattern, and having widely varying degrees of spacing between lines.

In such non-orthonormal systems, called *generalized coordinate* systems, the grid lines are numbered sequentially as 0,1,2,3,..., so one can always pinpoint one's location by specifying the number of the grid lines where one is located. Unlike orthonormal systems, however, these numbers do not directly reflect one's distance from the origin. If I am on a 2D flat surface at the intersection of generalized coordinate grid lines numbered 3 and 4, my coordinate location is (3,4), but I am not 3 physically measured meters (we will use meters as our measuring unit) from location (0,4). Depending on the spacing of the lines I could be 1 meter, 100 meters, a 1/2 meter, or any physical distance at all from (0,4). I can, of course, determine that distance experimentally by measuring it directly with a series of meter sticks laid end to end. But is there any way I can determine it analytically from my coordinate location values?

Before we begin our answer to that question, consider another situation where I wish to measure the speed of an object, but I can only count the number of grid lines the object passes by per second in my non-Cartesian system. How could I translate that speed into the speed I would measure using standard physical meter sticks instead of grid lines? The first speed (grid lines per second) is a *coordinate speed*. It is relative to the coordinate system we use. Different spacing of the grid lines means a different number of lines passed per second, and that means a different coordinate speed. The second speed (meters per second = standard meter sticks passed per second) is unique, however. It is the same regardless of the amount of squeezing or spreading out of our gridlines. It is a *physical speed*.

Each component of the velocity vector equals the speed in a given coordinate axis direction. If these components are in terms of grid lines per second, they are called *coordinate components* of velocity. If they are in terms of meter sticks per second, they are called *physical components.* For the special case where our coordinate grid is Cartesian (or Lorentzian), these components are identical, i.e., coordinate components equal the physical components we would measure.

Solutions to problems are typically found in terms of the coordinate system used, and hence are coordinate component solutions. But for non-orthonormal coordinates, we need to be able to relate those components to the values we would actually measure in experiment. In other words, we need to know how to find physical components from coordinate components.

Though widely used in applied mechanics[1],[2],[3],[4],[5], the method for doing this is rarely taught in general relativity courses and seemingly known to but a small minority of physicists. In fact, the author knows of only a single relativity text (Misner, Thorne, and Wheeler[6]) where the subject is even mentioned. This seeming oversight is probably due, in some measure, to the fact that invariant quantities are often those measured in experiments. For example, the time dilation experienced by a particle in a cyclotron is the proper time τ for that particle, and this is the same (invariant) for any coordinate system (orthonormal or not) used to calculate it. However, there are situations in which vector or tensor components (rather than invariant quantities) are measured, and since these can be different from the theoretically determined coordinate components, we need to be able to deduce the former from the latter.

This article provides an introduction to physical components, explains how they can be calculated, and illustrates their utility with two examples. One of these provides an answer to a question many students are commonly perplexed by. That is, "Is the speed of light in a non-inertial (general relativistic) frame equal to c?"

2 PHYSICAL COMPONENTS: THE SIMPLEST VIEW

Readers with marginal background in general relativity may wish to read the present more elementary and pedagogic section on physical components, and then skip to Section 5. Sections 3 and 4 use greater mathematical rigor to draw the same conclusions as Section 2 and cover more advanced topics such as tensor analysis.

We begin this section by refining the concept of generalized coordinates, i.e., those which may be other than Cartesian (or Lorentzian.) Note that generalized coordinates are by custom labeled with superscripts. That is, instead of three dimensions labeled x,y, and z, or x_1,x_2, and x_3, we use x^1, x^2, and x^3. Raising to a power is indicated using parenthesis, i.e., $(x^1)^2$.

To avoid confusion we deal solely with flat spaces until noted otherwise.

2.1 Generalized Coordinates

Consider the 2D flat space (no time dimension for the present) of Figure 1. Grid lines are numbered sequentially with integers. If we use a Cartesian coordinate system with X^1 and X^2 grid lines spaced one standard meter stick apart, then the distance ds between any two (infinitesimally close) points is found from the Pythagorean theorem

$$(ds)^2 = (dX^1)^2 + (dX^2)^2 \tag{1}$$

where dX^1 and dX^2 values equal physical (standard) meter stick distances in orthogonal directions. The reader should not be confused by the somewhat loose identification of the infinitesimal "d" in this discussion with the finite difference "Δ" in the finite size figures.

If, as in Figure 2, we change the grid line spacing to form a different, non-Cartesian, coordinate grid (x^1,x^2) for the same flat space, then distances between successive x^1 and x^2 grid lines are no longer one meter stick. So in the new grid, successive lines are still labeled as x^1=1, x^1=2, x^1=3, etc, but the distances between successive x^1 (and x^2) grid lines are no longer one meter. For example, the distance

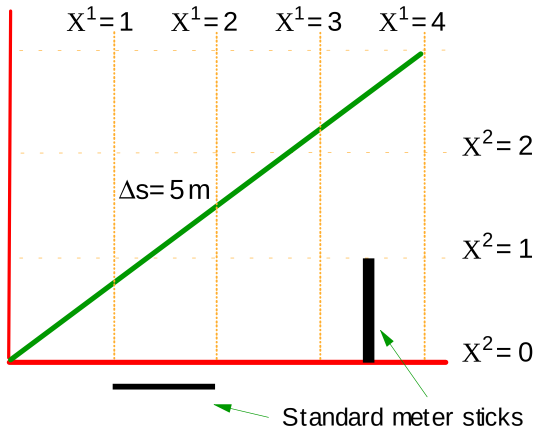


Fig 1 Cartesian Coordinate System

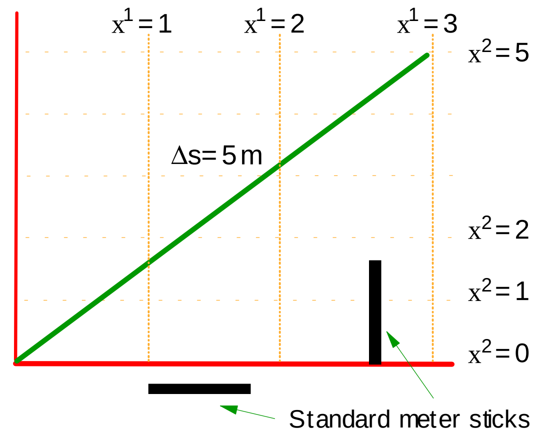


Fig 2 Generalized Coordinate System

between the x^1=1 and x^1=2 lines is not one meter, even though Δx^1=1 between those lines. So $dx^1$and dx^2 values equal number of grid lines between points, *not* physical meter stick distances (i.e., not number of meters) between the points, and the form of (1) for the new coordinates no longer holds. That is, for ds equal to the distance in meters between the same two points,

$$(ds)^2 \neq (dx^1)^2 + (dx^2)^2. \tag{2}$$

We therefore modify (1) to

$$(ds)^2 = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 \tag{3}$$

where g_{11} and g_{22} must be chosen such that (1) remains valid, i.e., g_{11} and g_{22} satisfy

$$g_{11}(dx^1)^2 = (dX^1)^2 \qquad g_{22}(dx^2)^2 = (dX^2)^2, \tag{4}$$

or more simply,

$$dX^1 = \sqrt{g_{11}}dx^1 \qquad dX^2 = \sqrt{g_{22}}dx^2. \tag{5}$$

In both (1) and (3) ds is the same distance (in meter stick lengths) between the same two points (ds is invariant between coordinate systems), and each expression is called the *line element* between the two points for that particular coordinate grid. The term generalized coordinates is used for such non-Cartesian coordinates because the coordinate grid (x^1,x^2) is completely arbitrary (general). The quantity g_{ij} (i,j= 1,2) is known as the *metric* of the coordinate grid. Note that for different generalized coordinate grids (x^1,x^2), there will be different values for g_{ij}.

2.2 Physical vs. Coordinate Components

dX^i in (5) are physically measured values (i.e., they are the number of meter sticks one would find physically by measuring in the X^1 and X^2 directions), and hence they are unique. The quantities dx^i, on the other hand, represent number of grid lines one would count off in the (x^1,x^2) system, and have no physical meaning, since they are different for each arbitrary coordinate grid. Hence, dX^i are called *physical components*[1],[6]; and dx^i, *coordinate components.*

The same relationship [see (5)] between components in Cartesian and generalized systems exists for vectors other than dx^i. For any vector expressed as v^iin a generalized coordinate system, we can find the equivalent physical component values $v^{\hat{i}}$ an experimentalist would actually measure via

$$v^{\hat{i}} = \sqrt{g_{\underline{ii}}}v^i, \tag{6}$$

where underlining implies no summation, and we have introduced the notation used by Misner, Thorne, and Wheeler[6] of carets over component indices to designate physical values.

2.3 Non-orthogonal Coordinate Grids

Note that if the coordinate axes in (1) are not orthogonal, then (1) and hence (3), get an extra cross term, i.e.,

$$(ds)^2 = (dX^1)^2 + (dX^2)^2 + 2dX^1dX^2\cos\theta = g_{ij}dx^idx^j \tag{7}$$

where repeated indices imply summation, θ is the angle between the X^1 and X^2 axes, and $g_{12} \neq 0$.

2.4 Non-flat Spaces

A small enough area on a curved surface such as that of a sphere appears approximately flat. Since we dealt only with differential lengths in relations (1) through (7), all of those relations remain valid (locally) on a curved surface. In particular, we find physical components (those measured experimentally) in the same manner, i.e., with (6), for any vector quantity in either a flat or non-flat space.

2.5 Spacetime

In special relativity, the four dimensional analogue of the Pythagorean theorem (1) is

$$(ds)^2 = -(cdT)^2 + (dX^1)^2 + (dX^2)^2 + (dX^3)^2 \tag{8}$$

where c is the speed of light, T is physical time, and ds is now the *spacetime interval,* an amalgam of both the physical spatial length and the physical time between two 4D *events.* Note that special relativity deals primarily with physical components and orthogonal grid lines in time and space, as in (8). If ds = 0, then the physical length in (8) (e.g., dX^1 where dX^2 = dX^3 = 0) divided by the physical time dT yields c, the physical speed of light. This result can be generalized, so if light travels between two events, then we always have ds =0 between those events. (Because of this, light is often said to travel a *null* path.)

In general relativity, one uses generalized 4D coordinate systems, and the analogue to (3) is

$$ds^2 = g_{00}(dx^0)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{9}$$

where $dx^0 = cdt$, $g_{00} < 0$, Greek indices are used to designate four dimensional (relativistic) quantities, and repeated indices imply summation over those indices. Here, dt represents *coordinate time*, which is an arbitrarily defined time (just as our spatial coordinate grids were arbitrarily defined) generally different from the physical time one would measure experimentally with standard clocks.

Use of coordinate time can simplify the solution of many problems. It can be visualized in a 3D connotation as a set of clocks that run at rates different than standard (physical) clocks and that are distributed at every point in our generalized 3D coordinate grid. In a 4D connotation, coordinate time can be visualized as a set of grid lines whose labels increment sequentially in the time direction, but which do not correspond to one second between grid lines.

Physical time is found from coordinate time in analogous fashion to (5) or (6), i.e.,

$$\text{physical time interval}{=}d\hat{t} = \sqrt{-g_{00}}dt; \qquad dx^{\hat{0}} = \sqrt{-g_{00}}dx^0. \tag{10}$$

Note that by substituting the coordinate component values of (5) [incorporating the third spatial dimension] and (10) into (9), we get (8), where the components in (8) are physical components for an

observer at rest in the generalized grid system. Since (8) represents a special relativistic, or Lorentz, coordinate system we can conclude the following.

In general relativity, for spacetime reference frames whose line elements can be cast in the form of (9)*, local physical components at any 4D point (event) are equal to those in a local co-moving Lorentz frame at the same 4D point.*[7]

Note further that for non-orthogonal axes, cross terms would be added to the line element (9) analogously to (7). If such cross terms are between time and space [e.g., a non-zero $g_{02}(cdt)(dx^2)$ term would be present in (9)] then time is not orthogonal to space[8] and, it can be shown that in such frames the above conclusion is not valid[9],[10].

3 PHYSICAL COMPONENTS: THE BASIS VECTOR VIEW

This and the following section are intended for readers having a reasonable foundation in general relativity. In particular, such readers should already be familiar with the concept of basis vectors and the metric identity

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j, \tag{11}$$

where $\mathbf{e}_\mathrm{i}$ is a generalized coordinate basis vector.

3.1 Unit and Generalized Basis Vectors

The displacement vector $d\mathbf{x}$ between two points in a 2D Cartesian coordinate system is

$$d\mathbf{x} = dX^1\hat{\mathbf{e}}_1 + dX^2\hat{\mathbf{e}}_2 \tag{12}$$

where the $\hat{\mathbf{e}}_i$ are unit basis vectors and dX^i are physical components. For the same vector $d\mathbf{x}$ expressed in a different, generalized, coordinate system we have different coordinate components $dx^\mathrm{i} \neq dX^\mathrm{i}$, but a similar expression

$$d\mathbf{x} = dx^1\mathbf{e}_1 + dx^2\mathbf{e}_2, \tag{13}$$

where the generalized basis vectors $\mathbf{e}_\mathrm{i}$point in the same directions as the corresponding unit basis vectors $\hat{\mathbf{e}}_i$, but are not equal to them. Hence, for $\hat{\mathbf{e}}_1$, we have

$$\hat{\mathbf{e}}_1 = \frac{\mathbf{e}_1}{|\mathbf{e}_1|} = \frac{\mathbf{e}_1}{\sqrt{\mathbf{e}_1 \cdot \mathbf{e}_1}} = \frac{\mathbf{e}_1}{\sqrt{g_{11}}} \tag{14}$$

where we used (11) to get the RHS. In general, we have

$$\hat{\mathbf{e}}_i = \frac{\mathbf{e}_i}{|\mathbf{e}_i|} = \frac{\mathbf{e}_i}{\sqrt{\mathbf{e}_{\underline{i}} \cdot \mathbf{e}_{\underline{i}}}} = \frac{\mathbf{e}_i}{\sqrt{g_{\underline{ii}}}} \tag{15}$$

where again, underlining implies no summation.

Substituting (15) into (12) and equating with (13), one obtains

$$dX^1 = \sqrt{g_{11}}dx^1 \qquad dX^2 = \sqrt{g_{22}}dx^2, \tag{16}$$

which is the same relationship between displacement physical and coordinate components as (5).

Consider a more general case of an arbitrary vector $\mathbf{v}$

$$\mathbf{v} = v^1\mathbf{e}_1 + v^2\mathbf{e}_2 = v^{\hat{1}}\hat{\mathbf{e}}_1 + v^{\hat{2}}\hat{\mathbf{e}}_2 \tag{17}$$

where, $\mathbf{e}_1$ and $\mathbf{e}_2$ here do not, in general, have to be orthogonal, $\mathbf{e}_i$ and $\hat{\mathbf{e}}_i$ point in the same direction for each index i, and as before, carets over component indices indicate physical components. Substituting (15) into (17), one readily obtains

$$v^{\hat{i}} = \sqrt{g_{\underline{ii}}}v^i, \tag{18}$$

which is the same as (6), and which we have shown here to be valid in both orthogonal and non-orthogonal systems.

In consonance with the above, many authors[11] note that the physical component of any vector in a given direction is merely the projection of that vector onto that direction, i.e., the inner product of the vector with a unit vector in the given direction. This is simply the component of the vector in a basis having a unit basis vector pointing in that direction.

If we apply this prescription by taking the dot product of $\hat{\mathbf{e}}_1$with $\mathbf{v}$ of (17), then use (15) and (11), we end up with (18) for i=1. This is obviously generalized to any index i.

As a further aid to those readers familiar with anholonomic coordinates (which superimpose unit basis vectors on a generalized coordinate grid), we reference Eringen's[12] comment ".. the anholonomic components .. of a ... vector ... are identical to [physical components]".

Following similar logic to that used in Section 2, one sees (18) is valid locally in curved, as well as flat, spaces, and can be extrapolated to 4D general relativistic applications. So, very generally, for a 4D vector u^μ

$$u^{\hat{i}} = \sqrt{g_{\underline{ii}}}u^i \qquad u^{\hat{0}} = \sqrt{-g_{00}}u^0, \tag{19}$$

where Roman sub and superscripts refer solely to spatial components (i.e. i = 1,2,3.)

3.2 Contravariant vs. Covariant Components

As the reader of this section is no doubt aware, coordinate displacement vector components dx^i(or dx^μin relativity theory) are *contravariant* components, as are the components of the vector expressed in (19). A similar derivation to that shown above carried out for *covariant* components leads to the following relationship between covariant coordinate components and covariant physical components

$$u_{\hat{i}} = \sqrt{g^{\underline{ii}}}u_i \qquad u_{\hat{0}} = \sqrt{-g^{00}}u_0. \tag{20}$$

3.3 Tensors

Considering second order tensors as open, or dyadic, products of vectors, with each tensor component having two basis vectors associated with it, one can use the general methodology of Sub-section 3.1 to derive the following relationships

$$T^{\hat{\mu}\hat{\upsilon}} = \sqrt{g_{\underline{\mu\mu}}}\sqrt{g_{\underline{\upsilon\upsilon}}}T^{\mu\upsilon} \qquad T_{\hat{\mu}\hat{\upsilon}} = \sqrt{g^{\underline{\mu\mu}}}\sqrt{g^{\underline{\upsilon\upsilon}}}T_{\mu\upsilon}. \tag{21}$$

As a first sample application of physical components, consider the 4D electromagnetic tensor $F^{\mu\nu}$ containing as components the electric and magnetic induction fields $\mathbf{E}$ and $\mathbf{B}$[13],[14]**.** Assume a cylindrical coordinate system (t, r, ϕ,z) with a field in the axial z direction measured to be B_m (subscript for "measured") and with no other electric or magnetic field components. The physical (measured) component of the magnetic induction in the z direction is B_m, and the corresponding term in the $F^{\mu\nu}$ tensor is the $\mu = r$, $\nu = \phi$ component. Hence,

$$F^{\hat{r}\hat{\phi}} = B_m. \tag{22}$$

The metric for a cylindrical coordinate system has g_{rr}= 1 and $g_{\phi\phi} = r^2$. Using these with (21), we then find the generalized tensor component, which should be used in any tensor analysis of the problem in cylindrical coordinates, to be

$$F^{r\phi} = \frac{F^{\hat{r}\hat{\phi}}}{\sqrt{g_{rr}}\sqrt{g_{\phi\phi}}} = \frac{B_m}{r}. \tag{23}$$

We note that tensor analysis must be carried out with coordinate components ($F^{\mu\nu}$ here), not physical components (i.e., not $F^{\hat{\mu}\hat{\upsilon}}$.) Fung[15] observes that "... physical components ... do not transform according to the tensor transformation laws and are not components of tensors."

Hence, at the beginning of an analysis we must first convert known measured (physical) component values to coordinate component values. Then we carry out the vector/tensor analysis. As a final step, in order to compare our results with experiment, we convert the coordinate component answer to physical component form.

4 ILLUSTRATING THE MOST GENERAL CASE

4.1 Visualizing basis vectors and physical components

The import of physical and coordinate values for both contravariant and covariant components can be visualized graphically. We start with two purely spatial sets of coordinate grid lines (x^1,x^2), which are quite arbitrary, and in this case, are not orthogonal. The contravariant component representation of a certain vector **V** is shown in Figure 3; the covariant representation for the same vector in Figure 4. Vector **V** is located at $x^1 = 2$ grid units, $x^2 = 10$ grid units and has magnitude and direction as shown.

Covariant basis vectors $\mathbf{e}_\mathrm{i}$ (Figure 3) for use with contravariant components are defined as, for example,

$$\mathbf{e}_2 = \left.\frac{\partial P}{\partial x^2}\right|_{x^1=2} = \frac{2 \text{ meter sticks}}{1 \text{ x}^2 \text{ grid unit}} = 2\hat{\mathbf{e}}_2 \tag{24}$$

where $\hat{\mathbf{e}}_2$is the unit vector in the direction of $\mathbf{e}_2$. Similar logic holds for $\mathbf{e}_{1,}$and we find in this example that $\mathbf{e}_1 = \hat{\mathbf{e}}_1$. The coordinate component of **V** in the $\mathbf{e}_2$ direction is $V^2 = 0$, the number of $\mathbf{e}_2$ basis vectors in that direction. The physical component in the same direction is $V^{\hat{2}} = 4$, the number of unit basis vectors $\hat{\mathbf{e}}_2$in that direction. This physical component is the same regardless of what grid we choose, whereas the coordinate component varies with the grid spacing.

$$V^{\hat{2}} = |\mathbf{e}_2| V^2 = \sqrt{\mathbf{e}_2 \cdot \mathbf{e}_2}\, V^2 = \sqrt{g_{22}} V^2 = \sqrt{2 \cdot 2}\, 2 = 4 \tag{25}$$

The physical component is the numerical value associated with the unit length basis vector $\hat{\mathbf{e}}_2$in the $\mathbf{e}_2$ direction.

As an aside, $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$ is obviously not diagonal, as $\mathbf{e}_i \cdot \mathbf{e}_j \neq 0$.

Contravariant basis vectors $\mathbf{e}^\mathrm{j}$ (called *one-forms* typically in general relativity) are defined, with a purpose in mind, by the relation

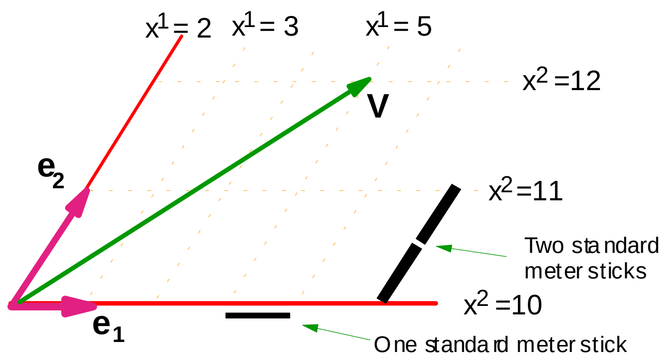
$$\mathbf{e}_i \cdot \mathbf{e}^j = \delta_i^j. \tag{26}$$

With (26) we get the contravariant basis vectors (or one-forms) shown in Figure 4. The covariant components V_1 and V_2 represent the number of contravariant (one-form) basis vectors $\mathbf{e}^1$ and $\mathbf{e}^2$, respectively, it takes along the $\mathbf{e}^1$ and $\mathbf{e}^2$ directions, respectively, to give us vector components along those directions that will vector sum to yield **V**. Of course, $\mathbf{e}^2$ and $\mathbf{e}_2$ are not aligned (as they would be for orthogonal grid lines). Neither are $\mathbf{e}^1$ and $\mathbf{e}_1$ aligned.

As with the contravariant component case, we can get determine any physical component, i.e., the number of unit vectors it will take along a given contravariant (one-form) basis vector direction for the vector component in that direction.

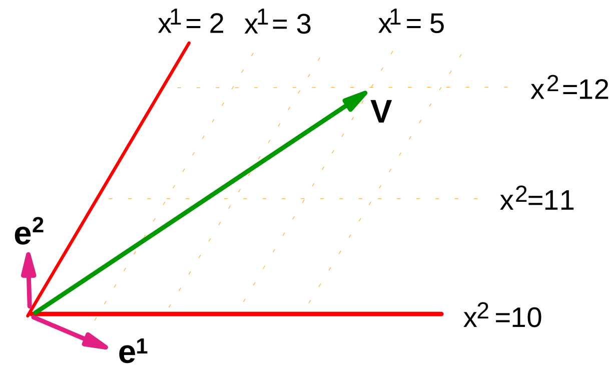
$$V_{\hat{2}} = |\mathbf{e}^2| V_2 = \sqrt{\mathbf{e}^2 \cdot \mathbf{e}^2}\, V_2 = \sqrt{g^{22}}\, V_2 \tag{27}$$

Note the contravariant physical component $V^{\hat{2}}$ of (25) does not equal the covariant physical component $V_{\hat{2}}$ of the above. The number of unit vectors needed in the $\mathbf{e}^2$ direction is not the same as that in the $\mathbf{e}_2$ direction.



$$\mathbf{V} = V^1\mathbf{e}_1 + V^2\mathbf{e}_2 = 3\mathbf{e}_1 + 2\mathbf{e}_2 = 3\hat{\mathbf{e}}_1 + 4\hat{\mathbf{e}}_2$$

Fig 3. Contravariant Components for Covariant Basis Vectors



$$\mathbf{V} = V_1\mathbf{e}^1 + V_2\mathbf{e}^2$$

Fig 4. Covariant Components for Contravariant Basis Vectors (One-forms)

4.2 Non-time-orthogonal Axes

Consider now the interesting case of spacetime around a star or black hole possessing angular momentum in which time is not orthogonal to space, i.e. time-space cross terms exist in the metric. In this case, instead of being spatial, our $\mathbf{e}_2$ direction in Figure 3 can be thought of as representing time. So wherever we have used a "2" sub or super script, now think of it as a "0". We shall refer to such a system as non-time-orthogonal (NTO). For this application we will also consider $\mathbf{V}$ as our (infinitesimal) displacement four-vector. That is $V^\mu = dx^\mu$ here.

Note that a clock fixed at $x^1 = 2$ travels a path in spacetime along the $\mathbf{e}_0$ (our old $\mathbf{e}_2$) direction, NOT along the $\mathbf{e}^0$(old $\mathbf{e}^2$) direction.

Of course we can now get coordinate values for our $\mu = 0$ contravariant and covariant components, and of course they will generally be different. However what about physical (standard) time? How many actual seconds (analogous to actual meter sticks) will a standard physical clock travel in spacetime if it is fixed in space at $x^1 = 2$? It is the physical component of the "0" component of $dx^0 = cdt$ (divided by c, of course.) That is, it is found from equation (25) with sub/superscripts $2 \rightarrow 0$. It is NOT the physical component of dx_0 (again divided by c) of equation (27), which gives a different numerical value.

Why can't we use either the contravariant or covariant physical component for standard (physically measured) time? Because $\mathbf{e}^2$ and $\mathbf{e}_2$ do not point in the same direction. Specifically the component $dx_{\hat{0}}$ does not represent the number of physical seconds along the $x^1 = \text{constant} = 2$ direction. It represents the physical component in the $\mathbf{e}^2$ direction, which does not represent a fixed spatial location within the given frame, but is actually a combination of both space and time.

Both contravariant and covariant representations are equivalent in the sense that both represent the same vector. For NTO frames, however, they are NOT equivalent in terms of how much of that vector is physically spacelike and how much is physically timelike.

So to represent time and space as they are actually measured with fixed clocks and fixed measuring meter sticks in such a frame, we have to use physical components associated with dx^μ, NOT those of dx_μ. Of course, for Lorentz frames it doesn't matter. And as far as physical components go, neither does it matter for Schwarzchild coordinates or any other time orthogonal (TO) frame. That is because in orthogonal coordinates, contravariant and covariant *physical* components are identical. For NTO frames, however, it definitely does matter whether one uses dx^μ or dx_μ , and when dealing with such frames one must proceed with caution.

In NTO frames, contravariant components of displacement correspond to true space and time directions; covariant displacement components do not. Therefore, four-velocity is, in the strictest sense, a contravariant vector since it equals $dx^\mu/\mathrm{d}\tau$.

4.3 General Case Conclusion

Although it is immaterial in Lorentz frames whether one uses contravariant or covariant components, and it is immaterial in any TO frame whether one uses contravariant or covariant *physical* components, it is critically important in NTO frames to use contravariant 4-vectors for displacement and velocity.

It will not matter in any case for invariants. They are not components of vectors or tensors, and will come out the same in any coordinates, in any kind of frame. However, for experimentally measured quantities that are components of a vector or tensor in an NTO frame, one must use physical components of the correct covariant or contravariant form.

Note that while physical values are not invariant between frames, within a given frame, they are unique. They equal what an observer in that frame would measure.

5 THE SPEED OF LIGHT

With an understanding of the difference between physical and coordinate components, one can now unravel the widespread confusion (in the author's experience at least) over the value of the speed of light in a gravitational field.

In a non-inertial (general relativistic) frame, the coordinate speed of light is found (see subsection 2.5) by setting $ds = 0$ in (9) and solving for the length in generalized coordinates (the coordinate length = number of spatial grid lines) that the light ray travels divided by the time in generalized coordinates (the coordinate time = number of temporal grid lines) it takes to travel, i.e., for (9) with $dx^2 = dx^3 = 0$,

$$\frac{dx^1}{dt} = \sqrt{\frac{-g_{00}}{g_{11}}}\, c = \frac{\text{(number of spatial grid lines traversed)}}{\text{(number of coordinate time units passed)}} \tag{28}$$

On the other hand, the physical speed of light is the physical length divided by the physical time,

$$\frac{\sqrt{g_{11}}dx^1}{\sqrt{-g_{00}}dt} = \; c = \frac{\text{(number of standard meter sticks traversed)}}{\text{(number of seconds on local standard clock)}}. \tag{29}$$

Note that (28) depends on the coordinate grid spacing, whereas (29) does not. We therefore emphasize the following.

In general relativity the local coordinate speed of light varies with the spacetime coordinate grid chosen, but for spacetime frames whose line elements can be cast in form (9)*, the local physical (experimentally measured) speed of light is always c.*

Most spacetime frames treated in general relativity can have their line elements cast in the form of (9). That is, they have no space-time cross terms and time is orthogonal to space. However, for frames that do have such cross terms[8], care must be taken. For details, see Klauber[9],[10].

6 SUMMARY

Physical components of vectors (or tensors) are the component values one would measure by experiment with standard physical instruments. They equal the components associated with unit basis vectors. Within a given frame they are unique.

Coordinate components are those one uses in vector/tensor analysis for a particular coordinate system. They equal the component values associated with generalized basis vectors. They are not unique and vary with the coordinate grid chosen.

Physical components and coordinate components are related by (19) for (contravariant) vectors and by (21) for second order tensors.

In doing tensor analysis one must first convert known physical (measured) component values to coordinate components, then carry out the vector/tensor analysis, and as a final step convert the coordinate components answer to physical components for comparison with experiment. For invariant quantities,

no conversion is necessary since they are the same for any coordinate system, they are not components of vectors or tensors, and they are already equivalent to physically measured values.

For time orthogonal frames:

1) the physical component in any direction equals the component value for the same direction found in a co-moving local Lorentz coordinate system,

2) the local physical speed of light is always c, and

3) the coordinate speed of light varies with the coordinate grid chosen, i.e., its numerical value is generally not equal to c.

References

[1] References [2],[3],[4], and [5] are a representative sample. The author considers the treatments of Malvern and Fung to be more transparent and more relevant to the present article. Note that in continuum mechanics, the term *curvilinear coordinates* is used for what relativity theorists call *generalized coordinates.* Further, basis vectors are typically designated with the symbol $\boldsymbol{g}_{\mathrm{i}}$ rather than $\mathbf{e}_{\mathrm{i}}$. And, depending on subtleties within certain intended applications, the definitions of covariant and mixed tensor physical components varies somewhat between authors.

[2] Lawrence E. Malvern, *Introduction to the Mechanics of a Continuous Medium* (Prentice-Hall, Englewood Cliffs, New Jersey, 1969). See Appendix I, Sec. 5, pp. 606-613.

[3] Y. C. Fung, *Foundations of Solid Mechanics* (Prentice-Hall, Inc., Englewood Cliffs, NJ, 1965). See pp. 52-53 and 111-115.

[4] A.Cemal Eringen, *Nonlinear Theory of Continuous Media,* (McGraw-Hill, NY, 1962). pp. 437-439.

[5] T.J. Chung, *Continuum Mechanics,* (Prentice Hall, Inc., Englewood Cliffs, NJ, 1988). pp. 40-53, 246-251.

[6] Charles W. Misner, Kip S. Thorne, and John A. Wheeler, *Gravitation* (Freeman, New York, 1973). Physical components are introduced on pg. 37, and used in many places throughout the text, e.g. pp. 821-822.

[7] This is the perspective from which physical components are introduced on pg. 37 of ref. [6].

[8] Such non-time-orthogonal systems actually occur in nature. Examples include spacetime around a star or black hole possessing angular momentum, and a rotating reference frame. For details on analyzing non-time-orthogonal frames, see refs. [9] and [10].

[9] Robert D. Klauber, "New perspectives on the relatively rotating disk and non-time-orthogonal reference frames", *Found. Phys. Lett.* 11(5) 405-443 (1998). gr-qc/0103076. See particularly, Section 5, pp. 429-434.

[10] Robert D. Klauber, "Non-time-orthogonal reference frames in the theory of relativity", gr-qc/0005121.

[11] For example, see ref [2], pp. 606-607; ref [3], pg. 111; ref. [4], pg. 65; ref. [5], pg. 437.

[12] A.Cemal Eringen, *Continuum Physics: Volume 1 Mathematics,* (Academic Press, NY and London, 1971). pp. 65-66.

[13] See ref. [6] pg. 74.

[14] John David Jackson, *Classical Electrodynamics,* (John Wiley & Sons, NY, 1975), Section 11.9, pg. 550. Sign differences from the present article are due to use of a different metric sign convention.

[15] See ref. [3], pg 53.